Homework 7

1. Thermal Averaging

   (a) Given your expression for the reduced reaction rate, λ, substitute the expressions for the cross-section involving the R- and S-factors and simplify.

   (b) For neutron-induced reactions, the R-factor is used. For roughly constant R-factors, we can expand R in a Taylor expansion in speed (v) or energy (E).

      i. Expand R in a Taylor expansion about $v = 0$ up to terms quadratic in v and find the resulting thermonuclear reaction rate λ, a function of temperature kT.

      ii. Expand R in a Taylor expansion about $E = 0$ up to terms quadratic in E and find the resulting thermonuclear reaction rate λ, a function of temperature kT.

   (c) For charge-induced reactions, the S-factor is used. If our S-factor is roughly constant, or slowly varying, then the main energy dependence of the integrand for the reaction rate is from exponential terms ($\exp\left[-2\pi\sqrt{Eg/E} - E/kT\right]$).

      i. Find the energy (E_0) where the argument of this exponential is maximum. This is known as the Gamow peak.

      ii. Taylor expand the argument of the exponential about this energy (E_0), up to quadratic terms. Rewriting the exponential as: $\exp\left[C_0 + C_1(E - E_0) + \frac{1}{2}C_2(E - E_0)^2\right]$. What are the coefficients C_i? It is convenient to write these in terms of E_0 and kT. It is useful to define: $C_2 = -\frac{1}{\Sigma_0^2}$. The quantity $\Delta_0 = 2\sqrt{2}\Sigma_0$ is called the "Full Width".

      iii. Is the ratio Σ_0/E_0 large or small when the temperature T is small? Does this mean the integrand is "wide" or "narrow" about the Gamow peak? The energy range where the integrand peaks (being defined by E_0 and Δ_0), defines the "Gamow window". Essentially telling us at what energies we need to know the reaction cross-section for particular astrophysical environments (specified by the temperature T).

      iv. Defining a new variable $z = \left(\frac{E - E_0}{\Sigma_0}\right)$, re-formulate the integral in terms of this new variable z. Assume the Taylor expansion approximation of the exponential is valid. What is the lower integration limit of the variable z? What is it as the temperature gets smaller?

      v. Motivated by the "narrowness" of the peak and the lower integration limit at small temperatures, let us evaluate the integral for an S-factor that is constant, assuming it is okay to use the zero-temperature lower integration limit for non-zero temperatures.

      vi. Re-express your reduced reaction rate in terms of the form:

$$\lambda = D_0(kT)^{power1} \exp\left[D_1(kT)^{power2}\right] \tag{1}$$

      What are the coefficients D_0 and D_1 and the powers, $power1$ and $power2$?

   (d) Narrow resonant reactions are very common in nuclear physics. Their cross-sections are well-described by:

$$\sigma(E) = \frac{\pi}{k^2} \frac{(\omega\gamma)\,\Gamma}{(E - E_r)^2 + \Gamma^2/4} \tag{2}$$

   where E_r is the resonance energy, Γ is the total width, and $\omega\gamma$ is the resonance strength. Note that this formula holds for both neutron- and charge-induced reactions.

      i. Evaluate the reduced reaction rate for a resonant cross-section in the limit that the total width Γ goes to zero. It will be useful to find a relation between the Lorentzian as $\Gamma \to 0$ and the Dirac delta function.

      ii. Re-express your reduced reaction rate in the above form, evaluating the new coefficients D_0 and D_1 and powers $power1$ and $power2$.